OPERATOR ALGEBRAS AND MEASURE PRESERVING AUTOMORPHISMS

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Introduction

A basic unsolved problem of ergodic theory is to classify the ergodic measure preserving automorphisms of the measure algebra of the unit interval, up to conjugacy. In this paper, it is shown that the theory of a single automorphism of this type is fully equivalent to the theory of a certain Banach algebra of operators on the Hilbert space $L^2[0, 1]$, in that the conjugacy problem for automorphisms is the same as the problem of unitary equivalence in this class of operator algebras (Theorem 1.8).

One question that arises is to what extent this method can be applied to more general groups of automorphisms, and this is taken up in section 2. Roughly, it is shown that the method is always workable if the underlying group is amenable (qua a discrete group); and for any non-amenable group of automorphisms the method always breaks down (Theorem 2.6).

These results were previously announced in [1].

1. The operator algebras $\mathcal{A}$ and $\mathcal{B}$

Consider the probability space consisting of the unit interval [0, 1], Borel sets, and Lebesgue measure. Let $\mathcal{H}$ be the Hilbert space $L^2[0, 1]$, and let M be the von Neumann algebra of all multiplications by bounded measurable functions, acting on $\mathcal{H}$. Lebesgue measure lifts to a countably additive probability measure m on the projections of M. A *-automorphism α of M is said to *preserve* m if $m \circ \alpha = m$ on the projections of M. α is *ergodic* if the only projections $P \in M$ for which $\alpha(P) = P$ are 0 and I. If β is another ergodic m-preserving *-automorphism, then α and β are *conjugate* if there exists a *-automorphism

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τ of M such that $\tau\circ\alpha=\beta\circ\tau$ (a standard argument, using ergodicity of α, shows that τ necessarily preserves m).

There exists a unitary operator U_α on $\mathcal{H}$ such that $\alpha(A)=U_\alpha A U_\alpha^*$, $A\in M$. For example, let f_0 be the constant function 1, and define U_α on the dense submanifold Mf_0 of $\mathcal{H}$ by U_α: $Af_0\to\alpha(A)f_0$. U_α is easily seen to be an isometry of Mf_0 onto itself, and its unitary extension implements α in the right way. If V is any other unitary operator such that $\alpha(A)=VAV^*$, then there exists a unitary $W\in M$ such that $V=U_\alpha W$. Indeed, $U_\alpha^* V$ is a unitary operator which commutes with M, and since M is maximal abelian, it follows that $W=U_\alpha^* V\in M$.

In the remainder of this section, α will be a fixed m-preserving ergodic *-automorphism of M, with U_α a unitary operator such that $\alpha=U_\alpha\cdot U_\alpha^*$ on M. Since $MU_\alpha=U_\alpha M$, the set $\mathcal{A}_0(\alpha)$ of finite sums $A_0+A_1U_\alpha+\ldots+A_nU_\alpha^n$, $n\geqslant 0$, $A_i\in M$, forms on algebra, and the set $\mathcal{B}_0(\alpha)$ of sums $A_mU_\alpha^m+\ldots+A_nU_\alpha^n$, $m\leqslant 0\leqslant n$, $A_i\in M$, forms a *-algebra. By the above remarks, neither $\mathcal{A}_0(\alpha)$ nor $\mathcal{B}_0(\alpha)$ depends on the particular choice of U_α. Let $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ be the respective closures in the operator norm. Clearly $M\subseteq\mathcal{A}(\alpha)\subseteq\mathcal{B}(\alpha)$, and of course $\mathcal{B}(\alpha)$ is a C^*-algebra.

While the powers α^n $(n\neq 0)$ of α are not in general ergodic, they are *freely-acting* in the sense that for every $n\neq 0$ and every projection $P\neq 0$ in M, there exists a subprojection $Q\in M$, $0\neq Q\leqslant P$, such that $\alpha^n(Q)\perp Q$ (see [6], p. 125). By choosing successively smaller projections, it follows that for every finite set F of integers such that $0\notin F$ and every projection $P\neq 0$ in M, there is a nonzero subprojection Q of P such that $\alpha^n(Q)\perp Q$ for all $n\in F$.

The following result appears in [10] p. 232, where, incidentally, the algebra $\mathcal{A}_0(\alpha)$ is shown to be an irreducible triangular subalgebra of the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. For completeness, we include a short proof.

LEMMA 1.1. *$A_mU_\alpha^m+\ldots+A_nU_\alpha^n=0$ implies $A_m=\ldots=A_n=0$.*

Proof. Suppose, to the contrary, that $A_mU_\alpha^m+\ldots+A_nU_\alpha^n=0$ and $A_m\neq 0$. Then $n\geqslant m+1$ necessarily, and $A_m+A_{m+1}U_\alpha+\ldots+A_nU_\alpha^{n-m}=0$. Choose a projection $P\neq 0$ in M such that $A_mQ\neq 0$ for every $Q\in M$ satisfying $0\neq Q\leqslant P$ (e.g., if A_m is multiplication by $f\in L^\infty[0,1]$, P can be multiplication by the characteristic function of the set $\{|f|\geqslant\varepsilon\}$, where ε is a positive number small enough that the set has positive measure). By free action, there exists $Q\in M$, $0\neq Q\leqslant P$, such that $\alpha^k(Q)\perp Q$ for all $k=1,2,\ldots,n-m$. For every $B\in M$, $1\leqslant k\leqslant n-m$, one has $QBU_\alpha^kQ=QB\alpha^k(Q)U_\alpha^k=Q\alpha^k(Q)BU_\alpha^k=0$. Hence,

$$A_mQ=QA_mQ=Q(A_m+A_{m+1}U_\alpha+\ldots+A_nU_\alpha^{n-m})Q=0,$$

and that contradicts the original choice of P.

Hence, one can define a linear map Φ of $\mathcal{B}_0(\alpha)$ onto M by

$$\Phi(\sum_k A_k U_\alpha^k) = A_0.$$

In the next few results, through Theorem 1.5, we develop some properties of this mapping. The simplest are these:

$$\begin{aligned} &\text{(i) } \Phi\circ\Phi = \Phi, \quad \Phi(I) = I \\ &\text{(ii) } \Phi(AT) = A\Phi(T), \quad \Phi(TA) = \Phi(T)A, \quad A \in M,\ T \in \mathcal{B}_0(\alpha) \\ &\text{(iii) } \Phi(T^*) = \Phi(T)^*, \quad T \in \mathcal{B}_0(\alpha) \\ &\text{(iv) } \Phi(MU_\alpha^n) = 0, \quad n \neq 0 \\ &\text{(v) } 0 \leqslant \Phi(T)^*\Phi(T) \leqslant \Phi(T^*T), \quad T \in \mathcal{B}_0(\alpha). \end{aligned} \tag{1.2}$$

Indeed, (i) through (iv) are obvious consequences of the definition, and (v) follows from them by expanding the inequality $0 \leqslant \Phi([T-\Phi(T)]^*[T-\Phi(T)]$.

If $\mathcal{B}_0(\alpha)$ were closed (norm), then (v) would imply that Φ is positivity-preserving, and *a fortiori* bounded. However, since the usual argument involves taking the positive square root of a positive operator, an operation requiring norm closure, we cannot conclude from (v) that Φ is bounded. One needs the following lemma, another consequence of free action.

LEMMA 1.3. *For every projection $P \neq 0$ in M, there exists a state ϱ of $\mathcal{B}(\alpha)$ such that $\varrho(P) = 1$ and $\varrho(MU_\alpha^n) = 0$, for all $n \neq 0$.*

Proof. Fix P. For every $N \geqslant 1$, let K_N be the set of all states ϱ of $\mathcal{B}(\alpha)$ such that $\varrho(P) = 1$ and $\varrho(MU_\alpha^n) = 0$, $1 \leqslant |n| \leqslant N$. Each K_N is a weak*-closed, and therefore compact, subset of the state space of $\mathcal{B}(\alpha)$, and $K_N \supseteq K_{N+1}$ for all N. By free action, there exists, for each N, a nonzero projection $Q_N \in M$ such that $0 \neq Q_N \leqslant P$ and $\alpha^k(Q_N) \perp Q_N$, $1 \leqslant |k| \leqslant N$. Choose any unit vector f in the range of Q_N, and put $\varrho_N(T) = (Tf, f)$, $T \in \mathcal{B}(\alpha)$. Clearly $\varrho_N(P) = 1$ and one has $\varrho_N(T) = \varrho_N(Q_N T Q_N)$, for all T. If $A \in M$ and $1 \leqslant |k| \leqslant N$, then

$$\varrho_N(AU_\alpha^k) = \varrho_N(Q_N A U_\alpha^k Q_N) = \varrho_N(Q_N \alpha^k(Q_N) A U_\alpha^k) = 0.$$

Thus $\varrho_N \in K_N$, and this shows that the K_N's have the finite intersection property. Hence $\bigcap K_N \neq \emptyset$, and any state in the intersection has all the required properties.

PROPOSITION 1.4. $\|\Phi(T)\| \leqslant \|T\|$, *for all* $T \in \mathcal{B}_0(\alpha)$.

Proof. Let $T \in \mathcal{B}_0(\alpha)$, and suppose $\|\Phi(T)\| > \|T\|$. Since $\Phi(T) \in M$, there exists a unitary operator $W \in M$ such that $\Phi(T)W \geqslant 0$. Now $\|\Phi(T)W\| = \|\Phi(T)\| \geqslant \|T\| + \varepsilon$, for sufficiently small $\varepsilon > 0$. So by spectral theory, there exists a projection $P \neq 0$ in M such that $\Phi(T)WP \geqslant (\|T\| + \varepsilon/2)P$. Now choose ϱ as in Lemma 1.3: $\varrho(P) = 1$ and $\varrho(MU_\alpha^n) = 0$, $n \neq 0$. By definition of Φ, it follows that $\varrho(S) = \varrho \circ \Phi(S)$ for every $S \in \mathcal{B}_0(\alpha)$. Hence,

$$\varrho(TWP) = \varrho \circ \Phi(TWP) = \varrho(\Phi(T)WP) \geqslant (\|T\| + \varepsilon/2)\varrho(P) = \|T\| + \varepsilon/2.$$

But this is impossible, since $|\varrho(TWP)| \leqslant \|TWP\| \leqslant \|T\|$, and this contradiction completes the proof.

One may now extend Φ to a bounded linear map of $\mathcal{B}(\alpha)$ onto M, by continuity. Properties (1.2) are valid for the extension, which we denote by the same letter Φ.

A positive linear map Φ_1 of one C^*-algebra into another is *faithful* if $\Phi_1(T^*T) = 0$ implies $T = 0$, for every T in the domain of Φ_1. We claim that Φ is faithful. For the proof, it suffices to produce another C^*-algebra $\mathcal{C}$, a positive faithful linear map ω of $\mathcal{C}$ into M, and a *-representation π of $\mathcal{C}$ such that $\pi(\mathcal{C}) = \mathcal{B}(\alpha)$ and $\Phi \circ \pi = \omega$. This gives the result, for if $T \in \mathcal{B}(\alpha)$ and $\Phi(T^*T) = 0$, then choose any $C \in \mathcal{C}$ such that $T = \pi(C)$. One has $\omega(C^*C) = \Phi \circ \pi(C^*C) = \Phi(\pi(C)^*\pi(C)) = \Phi(T^*T) = 0$. Since ω is faithful, $C = 0$, and finally $T = \pi(C) = 0$.

$\mathcal{C}$ is constructed as follows. Let Γ be the unit circle and let $\mathcal{C}_1$ be the collection of all functions F defined on Γ, taking values in the set $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$, and which are norm continuous:

$$\lim_{\theta \to \theta_0} \|F(e^{i\theta}) - F(e^{i\theta_0})\| = 0, \quad \text{for all } \theta_0 \in [0, 2\pi].$$

Endowed with the pointwise algebraic operations, pointwise involution, and the norm $|F| = \sup \|F(e^{i\theta})\|$, $\mathcal{C}_1$ becomes a C^*-algebra. The point evaluation $\pi(F) = F(1)$ is clearly a *-representation of $\mathcal{C}_1$ on $B(\mathcal{H})$. Now define

$$\omega(F) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta})\, d\theta, \quad F \in \mathcal{C}_1,$$

i.e., $\omega(F)$ is the operator in $B(\mathcal{H})$ satisfying

$$(\omega(F)f, g) = \frac{1}{2\pi} \int_0^{2\pi} (F(e^{i\theta})f, g)\, d\theta, \quad \text{for all } f, g \in \mathcal{H}.$$

Evidently, ω is a positive linear map of $\mathcal{C}_1$ into $B(\mathcal{H})$. If $\omega(F^*F) = 0$, then for every $f \in \mathcal{H}$,

$$\int_0^{2\pi} \|F(e^{i\theta})f\|^2 d\theta = \int_0^{2\pi} (F^*(e^{i\theta})F(e^{i\theta})f, f) = 2\pi(\omega(F^*F)f, f) = 0.$$

Since $e^{i\theta} \to \|F(e^{i\theta})f\|$ is a nonnegative continuous function on Γ, we have $F(e^{i\theta})f = 0$ for all θ. Hence, $F = 0$, and thus ω is faithful.

Now let $\mathcal{C}$ be the norm closure of the set of all functions of the form

$$F(e^{i\theta}) = A_m U_\alpha^m e^{im\theta} + \ldots + A_n U_\alpha^n e^{in\theta},$$

$m \leqslant 0 \leqslant n$, $A_i \in M$. $\mathcal{C}$ is clearly a C^*-subalgebra containing the identity. If F is of this form, then $\omega(F) = A_0$, $\pi(F) = A_m U_\alpha^m + \ldots + A_n U_\alpha^n$, and $\Phi \circ \pi(F) = A_0 = \omega(F)$. By norm continuity, $\Phi \circ \pi = \omega$ on $\mathcal{C}$. These formulas also show that $\pi(\mathcal{C}) \subseteq \mathcal{B}(\alpha)$ and $\omega(\mathcal{C}) \subseteq M$. Since π is a $*$-representation, $\pi(\mathcal{C})$ is a *closed* $*$-subalgebra of $\mathcal{B}(\alpha)$ which contains every finite sum $A_m U_\alpha^m + \ldots + A_n U_\alpha^n$, $A_i \in M$. Hence, $\pi(\mathcal{C}) = \mathcal{B}(\alpha)$, and the restrictions of π and ω to $\mathcal{C}$ have all the right properties. This proves the following:

THEOREM 1.5. *The extended map Φ is faithful on $\mathcal{B}(\alpha)$.*

We turn now to the Banach algebra $\mathcal{A}(\alpha)$. Clearly $M \subseteq \mathcal{A}(\alpha) \cap \mathcal{A}(\alpha)^*$. It is a key fact that this inclusion is actually equality: i.e., $\mathcal{A}(\alpha)$ determines M.

LEMMA 1.6. (i) *Φ is multiplicative on $\mathcal{A}(\alpha)$.*

(ii) *$\mathcal{A}(\alpha) \cap \mathcal{A}(\alpha)^* = M$.*

Proof. For (i), let $A, B \in M$ and let m and n be nonnegative integers. Then $A U_\alpha^m B U_\alpha^n = A\alpha^m(B) U_\alpha^{m+n}$, so that $\Phi(A U_\alpha^m B U_\alpha^n) = 0$ or AB according as $m+n>0$ or $m=n=0$. In either case, we have $\Phi(A U_\alpha^m B U_\alpha^n) = \Phi(A U_\alpha^m)\Phi(B U_\alpha^n)$. By summing on m and n, it follows that Φ is multiplicative on $\mathcal{A}_0(\alpha)$, and (i) follows by continuity.

For (ii), it suffices to show that $\mathcal{A}(\alpha) \cap \mathcal{A}(\alpha)^* \subseteq M$, and since $\mathcal{A}(\alpha) \cap \mathcal{A}(\alpha)^*$ is spanned by its self-adjoint elements, we need only show that every self-adjoint element of $\mathcal{A}(\alpha)$ is in M. Take $T = T^* \in \mathcal{A}(\alpha)$, and let $T_1 = T - \Phi(T)$. Then T_1 is self-adjoint, belongs to $\mathcal{A}(\alpha)$, and by (i),

$$\Phi(T_1^2) = \Phi(T_1)^2 = (\Phi(T) - \Phi \circ \Phi(T))^2 = 0.$$

By Theorem 1.5, $T_1 = 0$, and so $T = \Phi(T) \in M$.

The second key fact about $\mathcal{A}(\alpha)$ is that it determines α. This is a consequence of the following Lemma.

LEMMA 1.7. *Let V_1 and V_2 be two unitary operators in $\mathcal{A}(\alpha)$ such that*

(i) *$A \to V_i A V_i^*$ is a freely-acting automorphism of M, for $i = 1, 2$, and*

(ii) *V_2 belongs to the closed algebra generated by M and V_1.*

Then $V_2 V_1^* \in \mathcal{A}(\alpha)$.

Proof. First, we claim $\Phi(V_1) = \Phi(V_2) = 0$. Let $\mathcal{P}_1$ be the collection of all projections $P \in M$ such that $P \perp V_1 P V_1^*$. For $P \in \mathcal{P}_1$, we have

$$\Phi(V_1)P = P\Phi(V_1)P = P\Phi(V_1P) = P\Phi(V_1PV_1^*V_1) = PV_1PV_1^*\Phi(V_1) = 0.$$

Thus, $\Phi(V_1)=0$ on $P\mathcal{H}$. But by free action, the least upper bound of all projections in $\mathcal{D}_1$ is I. Hence, $\Phi(V_1)=0$. The same argument shows that $\Phi(V_2)=0$.

Since $V_1M=MV_1$, the set of finite sums $A_0+A_1V_1+...+A_NV_1^N$ ($A_i\in M$, $N\geqslant 0$) is dense in the closed algebra generated by M and V_1; so by (ii), there exists a sequence T_n of operators of this form such that $\|V_2-T_n\|\to 0$. Since $\Phi(V_2)=0$, $\Phi(T_n)\to 0$. Hence, if $T_n'=T_n-\Phi(T_n)$, then $T_n'\to V_2$. But $\Phi(V_1^k)=\Phi(V_1)^k=0$ for all $k\geqslant 1$ (this, by 1.6 (i)), so that if T_n has the form $A_0+A_1V_1+...+A_NV_1^N$, then T_n' looks like

$$A_1V_1+A_2V_1^2+...+A_nV_1^N=(A_1+A_2V_1+...+A_NV_1^{N-1})V_1.$$

In particular, $T_n'\in\mathcal{A}(\alpha)V_1$. Since $\mathcal{A}(\alpha)V_1$ is a closed subspace of $B(\mathcal{H})$ in the operator norm (V_1 is invertible), we conclude that $V_2\in\mathcal{A}(\alpha)V_1$, proving the lemma.

COROLLARY. *If V is a unitary operator in $\mathcal{A}(\alpha)$ such that $A\to VAV^*$ is a freely-acting automorphism of M and such that M and V generate $\mathcal{A}(\alpha)$ as a Banach algebra, then there exists a unitary $W\in M$ such that $V=U_\alpha W$.*

In particular, $\alpha(A)=VAV^*$, $A\in M$.

Proof. Applying Lemma 1.7 twice (once to the pair (V, U_α) and once again to (U_α, V)), we have $VU_\alpha^*\in\mathcal{A}(\alpha)$ and $(VU_\alpha^*)^*=U_\alpha V^*\in\mathcal{A}(\alpha)$. So if $W_1=VU_\alpha^*$, then $W_1\in\mathcal{A}(\alpha)\cap\mathcal{A}(\alpha)^*$, and $V=W_1U_\alpha=U_\alpha\cdot\alpha^{-1}(W_1)$, as required.

THEOREM 1.8. *Let α and β be ergodic m-preserving *-automorphisms of M. Then α and β are conjugate if, and only if, there exists a unitary operator T such that $T\mathcal{A}(\alpha)T^*=\mathcal{A}(\beta)$.*

Proof. For necessity, say τ is an m-preserving *-automorphism of M such that $\tau\circ\alpha=\beta\circ\tau$. There are a number of familiar methods that will produce the T that does the job. To sketch one, let f_0 be the constant function 1, and let ϱ be the vector state $\varrho(A)=(Af_0, f_0)$, $A\in M$. Then $\varrho(P)=m(P)$ if P is a projection in M; hence $\varrho\circ\tau(P)=\varrho(P)$. Since finite linear combinations of projections are norm dense in M, we have $\varrho\circ\tau(A)=\varrho(A)$, $A\in M$. Thus, the map $Af_0\to\tau(A)f_0$ ($A\in M$) extends to an isometry T of $[Mf_0]$ on $[\tau(M)f_0]=[Mf_0]$. Since f_0 is cyclic for M, T is unitary. Routine computation now shows that $TA=\tau(A)T$, $A\in M$, so that $TMT^*=M$. Moreover, $\tau\circ\alpha(A)=T\alpha(A)T^*=TU_\alpha AU_\alpha^*T^*$, and $\beta\circ\tau(A)=U_\beta TAT^*U_\beta^*$, $A\in M$. Since $\tau\circ\alpha=\beta\circ\tau$, we have $T^*U_\beta^*TU_\alpha\in M'=M$, so there exists a unitary $W\in M$ such that $TU_\alpha=U_\beta TW$. Thus,

$$TU_\alpha T^*=U_\beta TWT^*=U_\beta\tau(W)\in\mathcal{A}(\beta),\quad\text{and}\quad T^*U_\beta T=U_\alpha W^*\in\mathcal{A}(\alpha).$$

Since M and U_α (resp. U_β) generate $\mathcal{A}(\alpha)$ (resp. $\mathcal{A}(\beta)$), it follows that $T\mathcal{A}(\alpha)T^* \subseteq \mathcal{A}(\beta)$ and $T^*\mathcal{A}(\beta)T \subseteq \mathcal{A}(\alpha)$. Hence, $T\mathcal{A}(\alpha)T^* = \mathcal{A}(\beta)$, as asserted.

Now suppose, conversely, that T is a unitary operator such that $T\mathcal{A}(\alpha)T^* = \mathcal{A}(\beta)$. Hence, $\mathcal{A}(\beta)^* = T\mathcal{A}(\alpha)^*T^*$, so that by Lemma 1.6, $M = \mathcal{A}(\beta) \cap \mathcal{A}(\beta)^* = T(\mathcal{A}(\alpha) \cap \mathcal{A}(\alpha)^*)T^* = TMT^*$. Thus, $\tau(A) = TAT^*$ is a *-automorphism of M. Let V be the unitary operator $TU_\alpha T^* \in \mathcal{A}(\beta)$. The closed algebra generated by M and V is $T\mathcal{A}(\alpha)T^* = \mathcal{A}(\beta)$. Also, for $A \in M$, $VAV^* = TU_\alpha T^*ATU_\alpha^* T^* = \tau\circ\alpha\circ\tau^{-1}(A)$; since free-action is a conjugacy invariant, V satisfies all the hypotheses of the preceding corollary. Hence, $\beta(A) = VAV^* = \tau\circ\alpha\circ\tau^{-1}(A)$, completing the proof.

Remarks. Care must be exercised in closing the original algebra $\mathcal{A}_0(\alpha)$. Indeed, $\mathcal{A}_0(\alpha)$ contains the maximal abelian von Neumann algebra M, and it is easy to see that $\mathcal{A}_0(\alpha)$ has no closed invariant subspaces except $\{0\}$ and $\mathcal{H}$ ([10], p. 232). By the results of ([3]), any irreducible subalgebra of $B(\mathcal{H})$ which contains M is strongly dense in $B(\mathcal{H})$. In particular, the strong closure of $\mathcal{A}_0(\alpha)$ contains no information about α.

Nevertheless, $\mathcal{A}(\alpha)$ carries considerable structure. First, it is a norm-closed irreducible triangular subalgebra of $B(\mathcal{H})$ (not necessarily maximal, see [10]). Second, $\mathcal{A}(\alpha)$ can be realized as an algebra of operator-valued analytic functions in the unit disc. To sketch this very briefly, consider the C^*-algebra $\mathcal{C}$ constructed in the proof of Theorem 1.5. By making use of the classical Fejer kernel, one can show that $\mathcal{C}$ consists precisely of all norm-continuous functions $F\colon \Gamma \to B(\mathcal{H})$ such that $\int_0^{2\pi} F(e^{i\theta})e^{-in\theta}d\theta \in MU_\alpha^n$ for every n. Moreover, the *-representation π is actually a *-isomorphism. Thus, $\mathcal{B}(\alpha)$ is identified (non-spatially) with $\mathcal{C}$. $\mathcal{A}(\alpha)$ becomes the set of all $F \in \mathcal{C}$ for which $\int_0^{2\pi} F(e^{i\theta})e^{in\theta}d\theta = 0$, for $n \geq 1$, and there is a natural way of extending every such F analytically to the interior of the disc. Thirdly, one can associate with $\mathcal{A}(\alpha)$ a maximal subdiagaonal subalgebra (in the sense of [2]) of a II_1 factor. For this, regard $\mathcal{A}(\alpha)$ as a subalgebra of the function algebra $\mathcal{C}$. Let $\mathcal{K}$ be the Hilbert space of all $\mathcal{H}$-valued weakly measurable functions f such that $\int_0^{2\pi} \|f(e^{i\theta})\|^2 d\theta < \infty$. The map σ that associates with $F \in \mathcal{C}$ the operator

$$(\sigma(F)f)(e^{i\theta}) = F(e^{i\theta})f(e^{i\theta})$$

is a faithful *-representation of $\mathcal{C}$ such that the weak closure of $\sigma(\mathcal{C})$ is a hyperfinite II_1 factor acting on $\mathcal{K}$ (hyperfiniteness follows from [7], p. 576). Moreover, the ultraweak closure of $\sigma(\mathcal{A}(\alpha))$ is a maximal subdiagonal subalgebra of $\sigma(\mathcal{C})^-$ for which Jensen's inequality is valid (see [2]). This subdiagonal algebra also characterizes α up to conjugacy.

Finally, in a different direction, it follows from 1.8 that there is a very large number of unitarily inequivalent norm-closed irreducible triangular subalgebras of $B(\mathcal{H})$.

2. Generalizations

It may be of interest to know the extent to which Theorem 1.8 generalizes to more general groups of *-automorphisms. In particular, is there an analog of 1.8 for one-parameter groups? We shall not go into specific questions here, except to show that there is a natural boundary for the method. We prove that, while the results through 1.4 carry over routinely to arbitrary groups of *-automorphisms, 1.5 is true if, and only if, the group is amenable (qua a discrete group).

Let M be the multiplication algebra of section 1, let G be a group, and let $x\to\alpha_x$ be a homomorphism of G into the group of m-preserving *-automorphisms of M. We do not assume that the action is ergodic, but we do require that the action be *free* in the sense that for every $x\neq e$ and every projection $P\neq 0$ in M, there exists a nonzero subprojection Q of P (in M) such that $\alpha_x(Q)\perp Q$.

Let f_0 be the constant function 1. Note that there exists a unitary representation $x\to U_x$ of G on $\mathcal{H}$ such that $U_xAU_x^{-1}=\alpha_x(A)$ and $U_xf_0=f_0$, for every $x\in G$. Indeed, arguing as in the first paragraphs of § 1, the map

$$U_x: Af_0\to\alpha_x(A)f_0,\quad A\in M,$$

extends to a unitary operator having the properties $U_xf_0=f_0$ (by definition), and $U_xA=\alpha_x(A)U_x$, $A\in M$. One has, moreover,

$$U_xU_y: Af_0\to\alpha_x\circ\alpha_y(A)f_0=\alpha_{xy}(A)f_0,$$

so that $U_xU_y=U_{xy}$. The property $U_xf_0=f_0$ will enter in an essential way to the proof of Theorem 2.6.

Now if $x\to A_x$ is an M-valued function on G such that $A_x=0$ for all but finitely many x, we can form the operator $\sum_x A_xU_x$; since $MU_x=U_xM$ for all x, the set of all operators of this form is a *-algebra, whose norm closure we call $\mathcal{B}$. Note that $\sum A_xU_x=0$ implies $A_x=0$ for all x. Indeed, for fixed x_0, let F be the finite set $\{xx_0^{-1}: x\neq x_0 \text{ and } A_x\neq 0\}$, and let $\mathcal{D}$ be the family of projections $P\in M$ such that $\alpha_t(P)\perp P$ for $t\in F$. Arguing as in Lemma 1.1, we have, for $P\in\mathcal{D}$,

$$A_{x_0}P=PA_{x_0}P=P\sum_x A_xU_{xx_0^{-1}}P=P(\sum A_xU_x)U_{x_0}^*P=0.$$

By free action, $\mathrm{LUB}\,\mathcal{D}=I$, hence $A_{x_0}=0$. Thus, one can define Φ exactly as before:

$$\Phi(\sum A_xU_x)=A_e,$$

e being the identity of G.

PROPOSITION 2.1. *$\|\Phi(T)\| \leqslant \|T\|$ for every $T = \sum A_x U_x$. Thus Φ extends by continuity to $\mathcal{B}$. The extension has the following properties:*

(i) $\Phi \circ \Phi = \Phi, \quad \Phi(I) = I$

(ii) $\Phi(AT) = A\Phi(T), \quad \Phi(TA) = \Phi(T)A, \quad A \in M,\ T \in \mathcal{B}$

(iii) $\Phi(T^*) = \Phi(T)^*, \quad T \in \mathcal{B}$

(iv) $\Phi(MU_x) = 0, \quad x \neq e$

(v) $0 \leqslant \Phi(T)^*\Phi(T) \leqslant \Phi(T^*T)$.

Proof. The argument is a trivial alteration of that already given in § 1. First, arguing as in 1.3, one shows that for each $P \in M$, $P \neq 0$, there exists a state ϱ of $\mathcal{B}$ such that $\varrho(P) = 1$ and $\varrho(MU_x) = 0$, $x \neq e$. The proof of 1.4 can then be repeated verbatim to show that $\|\Phi\| \leqslant 1$. Finally, (i) through (v) are valid for the same reasons as 1.2.

From here on, we regard Φ as extended to the C^*-algebra $\mathcal{B}$. (v) shows that Φ is positive, and it makes sense therefore to ask if Φ is faithful.

A (discrete) group G is said to be *amenable* if there exists a finitely additive probability measure μ on the field of all subsets of G such that $\mu(xE) = \mu(E)$ for all $x \in G$, $E \subseteq G$. When such a μ exists, it is called a *mean* ([9], pp. 230–245). Solvable groups are amenable, as are locally finite groups. The free group on $n \geqslant 2$ generators is not amenable. An important characterization of amenability was found by Følner [8], and his proof has recently been greatly simplified by Namioka [11]. Følner's condition is that G is amenable if and only if for every finite subset $F \subseteq G$ and $\varepsilon > 0$, there is a finite subset $E \subseteq G$ such that

$$\left| \frac{|xE \cap E|}{|E|} - 1 \right| \leqslant \varepsilon,$$

for all $x \in F$ ($|\cdot|$ denotes "number of elements in"). We shall need a preliminary lemma characterizing amenability in a different way.

Let $C_{00}(G)$ be the set of complex valued functions on G having finite support, and let l_x be the left regular representation of G in $l^2(G)$ ($l_x\xi(y) = \xi(x^{-1}y)$, $\xi \in l^2(G)$). For every $\lambda \in C_{00}(G)$, we can form the operator $T = \sum_x \lambda(x) l_x$ on $l^2(G)$; λ is of *positive type* if $T \geqslant 0$. We believe the following lemma is known; e.g., it is proved in ([5], p. 319) that condition (iii) is equivalent to a slightly strengthened version of (ii). However, we know of no reference in the literature that relates these conditions to amenability. For completeness, then, we have included a proof.

LEMMA 2.2. *The following are equivalent, for any discrete group G:*

(i) G is *amenable.*

(ii) *If* $\lambda \in \mathbf{C}_{00}(G)$ *is of positive type, then* $\sum_x \lambda(x) \geqslant 0$.

(iii) *For every finite subset* F *of* G *and* $\varepsilon > 0$, *there exists a unit vector* $\xi \in l^2(G)$ *such that* $|(l_x\xi, \xi) - 1| \leqslant \varepsilon$ *for all* $x \in F$.

Proof. (i) *implies* (ii): Let $\lambda \in \mathbf{C}_{00}(G)$ be such that $\sum \lambda(x) l_x \geqslant 0$, and let F be a finite set such that $\lambda = 0$ off F. Fix $\varepsilon > 0$. By the Følner–Namioka theorem, there exists a finite subset $E \neq \emptyset$ of G such that

$$\left| \frac{|xE \cap E|}{|E|} - 1 \right| \leqslant \varepsilon$$

for all $x \in F$. Put $\xi(t) = |E|^{-\frac{1}{2}} \chi_E(t)$. Then $\|\xi\|_2 = 1$, and for every $x \in G$,

$$(l_x\xi, \xi) = |E|^{-1} \sum_t \chi_E(x^{-1}t)\, \chi_E(t) = \frac{|xE \cap E|}{|E|}.$$

Hence, $|(l_x\xi, \xi) - 1| \leqslant \varepsilon$ whenever $x \in F$. Now

$$\left|\sum \lambda(x)\,(l_x\xi, \xi) - \sum \lambda(x)\right| \leqslant \sum |\lambda(x)| \cdot |(l_x\xi, \xi) - 1| \leqslant \varepsilon \sum |\lambda(x)|.$$

By hypothesis, $\sum \lambda(x)(l_x\xi, \xi) \geqslant 0$, and since ε is arbitrary, this shows that the distance between the complex number $\sum \lambda(x)$ and the nonnegative real axis can be made as small as we please. Thus $\sum \lambda(x) \geqslant 0$.

(ii) *implies* (iii): Suppose (iii) fails. Then there exist $x_1, \ldots, x_n \in G$ and $\varepsilon > 0$ such that $\max_i |(l_{x_i}\xi, \xi) - 1| \geqslant \varepsilon$, for every unit vector $\xi \in l^2(G)$. Let Ω be the set of all vector states ω_ξ, $\|\xi\| = 1$, defined on the von Neumann algebra $\mathcal{L}$ generated by $\{l_x : x \in G\}$. Since $\mathcal{L}$ has a separating vector (e.g., the characteristic function of $\{e\}$), every normal state of $\mathcal{L}$ is a vector state ([4], p. 233). It follows at once that Ω is convex. Let K be the following subset of $\mathbf{C}^n$:

$$K = \{(\varrho(l_{x_1}), \ldots, \varrho(l_{x_n})) : \quad \varrho \in \Omega\}.$$

K is convex, and the first lines of the proof show that $(1, 1, \ldots, 1)$ does not belong to the closure of K. Hence, there exists a linear functional F on $\mathbf{C}^n$ and a real number r such that $\operatorname{Re} F(K) \geqslant r > \operatorname{Re} F(1, 1, \ldots, 1)$. If $F(z_1, \ldots, z_n) = \sum a_k z_k$, define the operator T on $l^2(G)$ by

$$T = \sum c_k l_{x_k} + \sum \bar{c}_k l_{x_k}^*.$$

Then for every $\xi \in l^2(G)$, $\|\xi\| = 1$, one has

$$(T\xi, \xi) = \operatorname{Re} F((l_{x_1}\xi, \xi), \ldots, (l_{x_n}\xi, \xi)) \geqslant r;$$

moreover, $r > \sum c_k + \sum \bar{c}_k$. Hence, $T - rI \geqslant 0$ and $\sum c_k + \sum \bar{c}_k - r < 0$, contradicting (ii).

(iii) *implies* (i): Let $x_1, ..., x_n \in G$, $\varepsilon > 0$. Choose a unit vector $\xi \in l^2(G)$ such that

$$|(l_{x_i}\xi, \xi) - 1| \leqslant \varepsilon, \ 1 \leqslant i \leqslant n.$$

For $E \subseteq G$, put
$$\mu(E) = \sum_{x \in E} |\xi(x)|^2.$$

Then μ is a finitely (in fact, countably) additive probability measure. We claim

$$|\mu(x_i E) - \mu(E)| \leqslant 2(2\varepsilon)^{\frac{1}{2}},$$

for every $E \subseteq G$ and $i = 1, 2, ..., n$. Indeed, $\mu(E) = \sum_x \chi_E(x)\xi(x)\bar{\xi}(x) = (\chi_E\xi, \xi)$, and for every $y \in G$,

$$\mu(yE) = \sum_E |\xi(y^{-1}x)|^2 = (\chi_E l_y \xi, l_y \xi).$$

Hence
$$\begin{aligned} |\mu(yE) - \mu(E)| &= |(\chi_E\xi, \xi) - (\chi_E\xi, l_y\xi) + (\chi_E\xi, l_y\xi) - (\chi_E l_y\xi, l_y\xi)| \\ &\leqslant |(\chi_E\xi, \xi - l_y\xi)| + |(\chi_E(\xi - l_y\xi), l_y\xi)| \\ &\leqslant \|\chi_E\xi\| \cdot \|\xi - l_y\xi\| + \|\chi_E(\xi - l_y\xi)\| \cdot \|l_y\xi\| \\ &\leqslant 2\|\xi - l_y\xi\|. \end{aligned}$$

But $\|\xi - l_{x_i}\xi\|^2 = \|\xi\|^2 - 2\operatorname{Re}(\xi, l_{x_i}\xi) + \|l_{x_i}\xi\|^2 = 2(1 - \operatorname{Re}(l_{x_i}\xi, \xi)) \leqslant 2\varepsilon$, $i = 1, ..., n$. Hence, $|\mu(x_i E) - \mu(E)| \leqslant 2(2\varepsilon)^{\frac{1}{2}}$, as asserted.

For $x_1, ..., x_m \in G$ and $\varepsilon > 0$, let $\mathcal{J}(x_1, ..., x_m; \varepsilon)$ be the collection of all finitely additive probability measures μ on G such that $|\mu(x_i E) - \mu(E)| \leqslant \varepsilon$ for every $E \subseteq G$ and $i = 1, ..., m$. We have just shown that the $\mathcal{J}$'s have the finite intersection property. Since the space of finitely additive probability measures is compact (in the topology of "pointwise" convergence at every subset of G), there is an element $\nu \in \bigcap \mathcal{J}$. ν is clearly a mean. That completes the proof.

Remarks. There are a number of other characterizations of (i)–(iii) that are more useful in different situations. For example, one may add to the list:

(iv) Every irreducible representation of G is weakly contained in the left regular representation. See ([5], p. 319).

We return now to the main problem, G, U_x, α_x, and Φ are as above. We shall construct another C^*-algebra $\mathcal{C}$ and a faithful positive linear map $\omega: \mathcal{C} \to M$. We then define a $*$-homomorphism π_0 of a dense $*$-subalgebra of $\mathcal{C}$ into $\mathcal{B}$, such that $\Phi \circ \pi_0 = \omega$, and we shall analyze what happens when one tries to extend π_0 to $\mathcal{C}$.

Form the Hilbert space $\mathcal{H} \otimes l^2(G)$. If $A, B \in M$, $x, y \in G$, then

$$(AU_x \otimes l_x)(BU_y \otimes l_y) = AU_xBU_y \otimes l_{xy} = A\alpha_x(B)U_{xy} \otimes l_{xy},$$

and
$$(AU_x \otimes l_x)^* = (AU_x)^* \otimes l_{x^{-1}} = \alpha_{x^{-1}}(A^*)U_{x^{-1}} \otimes l_{x^{-1}}.$$

Thus, the set $\mathfrak{C}_0$ of all finite sums $\sum_x A_x U_x \otimes l_x$ forms a *-algebra with identity. Let $\mathfrak{C}$ be the norm closure of $\mathfrak{C}_0$. Let ξ_e be the characteristic function of $\{e\}$, regarded as an element of $l^2(G)$.

LEMMA 2.3. *There exists a faithful positive linear map* $\omega: \mathfrak{C} \to M$ *such that* $(\omega(C)f, g) = (C(f \otimes \xi_e), g \otimes \xi_e)$, *for every* $f, g \in \mathcal{H}$. *One has*

$$\omega(\sum_x A_x U_x \otimes l_x) = A_e$$

for every function $x \in G \to A_x \in M$ *such that* $A_x = 0$ *for all but finitely many* x.

Proof. For every fixed $C \in \mathfrak{C}$, $[f, g] = (C(f \otimes \xi_e), g \otimes \xi_e)$ defines a bilinear form on $\mathcal{H} \times \mathcal{H}$ such that

$$|[f, g]| \leq \|C\| \cdot \|f \otimes \xi_e\| \cdot \|g \otimes \xi_e\| = \|C\| \cdot \|f\| \cdot \|g\|.$$

By a familiar lemma of Riesz, there exists a unique operator $\omega(C) \in B(\mathcal{H})$ such that $(\omega(C)f, g) = (C(f \otimes \xi_e), g \otimes \xi_e)$. ω is clearly linear, positive, norm-depressing, and $\omega(I) = I$. If $A_x = 0$ for all but finitely many x, then

$$(\sum_x A_x U_x \otimes l_x (f \otimes \xi_e), g \otimes \xi_e) = \sum_x (A_x U_x f \otimes l_x \xi_e, g \otimes \xi_e) = \sum_x (A_x U_x f, g)\,(l_x \xi_e, \xi_e).$$

Since $l_x \xi_e \perp \xi_e$ for $x \neq e$, the right side reduces to $(A_e f, g)$. Thus, $\omega(\sum A_x U_x \otimes l_x) = A_e$.

It remains to show that ω is faithful. Take $C \in \mathfrak{C}$ and suppose $\omega(C^* C) = 0$. Then

$$\|C(f \otimes \xi_e)\|^2 = (C^* C(f \otimes \xi_e), f \otimes \xi_e) = (\omega(C^* C)f, f) = 0,$$

so that $C(f \otimes \xi_e) = 0$ for every $f \in \mathcal{H}$. Now let r_x be the right regular representation of G in $l^2(G)$ $(r_x \eta(y) = \eta(yx),\ \eta \in l^2(G))$. Then $r_x l_y = l_y r_x$, so that each operator $I \otimes r_x$ commutes with $\mathfrak{C}$. Thus,

$$C(f \otimes r_x \xi_e) = C(I \otimes r_x) f \otimes \xi_e = I \otimes r_x C(f \otimes \xi_e) = 0,$$

for every $f \in \mathcal{H}$, $x \in G$. But $\{r_x \xi_e : x \in G\}$ is fundamental in $l^2(G)$; hence $\{f \otimes r_x \xi_e : f \in \mathcal{H},\ x \in G\}$ is fundamental in $\mathcal{H} \otimes l^2(G)$. Thus $C = 0$, and that completes the proof.

We claim that $\sum A_x U_x \otimes l_x = 0$ implies $A_x = 0$ for all x. Indeed, for every $x_0 \in G$,

$$\sum_x \alpha_{x_0}(A_x)\, U_{x_0 x} \otimes l_{x_0 x} = (U_{x_0} \otimes l_{x_0}) \sum A_x U_x \otimes l_x = 0.$$

Applying ω, we have $\alpha_{x_0}(A_{x_0^{-1}}) = 0$, hence $A_{x_0^{-1}} = 0$. One can now define $\pi_0: \mathfrak{C}_0 \to \mathcal{B}$ by

$$\pi_0(\sum A_x U_x \otimes l_x) = \sum A_x U_x.$$

π_0 is clearly a *-homomorphism of $\mathfrak{C}_0$ on a dense *-subalgebra of $\mathcal{B}$, and by definition of

ω and Φ, we have $\Phi \circ \pi_0 = \omega$ on $\mathcal{C}_0$. The question now is whether π_0 is bounded, and therefore extendable to $\mathcal{C}$.

LEMMA 2.4. *π_0 is bounded if, and only if, Φ is faithful.*

Proof. If π_0 is bounded, it extends by continuity to a *-representation π of $\mathcal{C}$ in $\mathcal{B}$ such that $\Phi \circ \pi = \omega$. $\pi(\mathcal{C})$ is closed, and it contains $\pi_0(\mathcal{C}_0)^- = \mathcal{B}$. Hence, $\pi(\mathcal{C}) = \mathcal{B}$. The argument used in § 1 now shows that Φ is faithful.

Conversely, assume Φ is faithful. Let $T \in \mathcal{C}_0$, $\|T\| \leqslant 1$. We will prove that $\pi_0(T^*T) \leqslant I$. It follows that π_0 is bounded for $\|\pi_0(T)\|^2 = \|\pi_0(T)^*\pi_0(T)\| = \|\pi_0(T^*T)\| \leqslant 1$.

Let $H = T^*T \in \mathcal{C}_0$. Then $0 \leqslant H \leqslant I$. Consider the abelian C^*-algebra generated by I and the positive operator $\pi_0(H)$. By spectral theory, we can regard this as the algebra $\mathbf{C}(X)$ of all continuous functions on a compact Hausorff space X. Let f be the functional representative of $\pi_0(H)$. If $\pi_0(H) \leqslant I$ fails, then there is a number $r > 1$ such that the open set $U = \{\gamma \in X : f(\gamma) > r\}$ is nonvoid. Let $g \in \mathbf{C}(X)$ be such that $0 \leqslant g \leqslant 1$, $g = 1$ somewhere in U, and $g = 0$ on $X - U$ (if $U = X$, we drop the last requirement on g). Then $g(\gamma) \neq 0$ implies $f(\gamma) \geqslant r$, so that $gf^n \geqslant r^n g$ for $n = 1, 2, \ldots$. Let A be the operator corresponding to g. Then $A \neq 0$, $0 \leqslant A \leqslant I$, $A\pi_0(H) = \pi_0(H)A$, $A \in \mathcal{B}$, and $A\pi_0(H^n) = A\pi_0(H)^n \geqslant r^n A$ for all $n \geqslant 1$.

Since Φ is assumed faithful, $\Phi(A) \neq 0$. Let ϱ_1 be any state of $\mathcal{B}$ such that $\varrho_1(\Phi(A)) \neq 0$, and put $\varrho = \varrho_1 \circ \Phi$. Then ϱ is a state, $\varrho \circ \Phi = \varrho$, and $\varrho(A) \neq 0$. Using the Schwarz inequality and $\Phi \circ \pi_0 = \omega$, we have, for $n \geqslant 1$:

$$\varrho(A\pi_0(H^n))^2 \leqslant \varrho(A^2)\varrho(\pi_0(H^n)^2) \leqslant \varrho(I)\varrho(\pi_0(H^n)^2)$$
$$= \varrho(\pi_0(H^{2n})) = \varrho \circ \Phi(\pi_0(H^{2n})) = \varrho \circ \omega(H^{2n}).$$

But $$H^{2n} \leqslant I,\ \omega(H^{2n}) \leqslant \omega(I) = I, \text{ and } \varrho \circ \omega(H^{2n}) \leqslant \varrho(I) = 1.$$

Hence, $$\varrho(A\pi_0(H^n)) \leqslant 1.$$

At the same time, $r^n A \leqslant A\pi_0(H^n)$, and so $r^n \varrho(A) \leqslant \varrho(A\pi_0(H^n)) \leqslant 1$. Since n is arbitrary, the last inequality implies that $\varrho(A) = 0$, a contradiction.

Before giving the main result, we have to construct one more mapping. If $\mathcal{F}$ is a family of operators, $[[\mathcal{F}]]$ will denote the C^*-algebra generated by $\mathcal{F}$. Let f_0 be a unit vector in $\mathcal{H}$ such that $U_x f_0 = f_0$ for all x (e.g., the constant function 1 will do).

LEMMA 2.5. *There exists a *-isomorphism σ of $[[U_x \otimes l_x : x \in G]]$ onto $[[l_x : x \in G]]$ such that $(\sigma(C)\xi, \eta) = (C(f_0 \otimes \xi), f_0 \otimes \eta)$ for every $C \in [[U_x \otimes l_x : x \in G]]$ and every $\xi, \eta \in l^2(G)$.*

If $\lambda \in \mathbf{C}_{00}(G)$, then $\sigma(\sum_x \lambda(x) U_x \otimes l_x) = \sum \lambda(x) l_x$.

Proof. Applying the Schwarz lemma as in Lemma 2.3, there exists, for each $C \in$

$[[U_x \otimes l_x : x \in G]]$, an operator $\sigma(C) \in B(l^2(G))$ such that $(\sigma(C)\xi, \eta) = (C(f_0 \otimes \xi), f_0 \otimes \eta)$ for all ξ, η. σ is clearly a positive linear map, and $\sigma(I) = I$ follows because $\|f_0\| = 1$.

For each $x \in G$, $\xi, \eta \in l^2(G)$, we have

$$(\sigma(U_x \otimes l_x)\xi, \eta) = (U_x \otimes l_x(f_0 \otimes \xi), f_0 \otimes \eta) = (U_x f_0, f_0)(l_x \xi, \xi) = (l_x \xi, \xi).$$

It follows that $\sigma(U_x \otimes l_x) = l_x$, and by taking finite sums, that $\sigma(\sum \lambda(x) U_x \otimes l_x) = \sum \lambda(x) l_x$, for every $\lambda \in \mathbf{C}_{00}(G)$. From this formula, it is evident that σ is multiplicative and *-preserving on the dense *-subalgebra of finite sums $\sum \lambda(x) U_x \otimes l_x$; by continuity, then, σ is a *-homomorphism. The range of σ is the closure of all finite sums $\sigma(\sum \lambda(x) U_x \otimes l_x) = \sum \lambda(x) l_x$, and hence σ is onto $[[l_x : x \in G]]$.

It remains to show that $\sigma(C) = 0$ implies $C = 0$. Let ξ_e be the characteristic function of e, qua an element of $l^2(G)$. We have used before the fact that $\varrho(T) = (T\xi_e, \xi_e)$ is a faithful state of $[[l_x : x \in G]]$. Now we claim $\omega(C) = \varrho \circ \sigma(C) I$, for every $C \in [[U_x \otimes l_x : x \in G]]$. Indeed, $\omega(U_x \otimes l_x) = 0$ or I according as $x \neq e$ or $x = e$, and $\varrho \circ \sigma(U_x \otimes l_x) = \varrho(l_x) = (l_x \xi_e, \xi_e)$. The claim follows because $l_x \xi_e \perp \xi_e$ whenever $x \neq e$. Now let $C \in [[U_x \otimes l_x : x \in G]]$ be such that $\sigma(C) = 0$. Then $\omega(C^* C) = \varrho \circ \sigma(C^* C) = \varrho(\sigma(C)^* \sigma(C)) = 0$. As ω is faithful, $C = 0$ as required.

THEOREM 2.6. *Φ is faithful if, and only if, G is amenable.*

Proof. First, suppose G is amenable. By Lemma 2.4, it suffices to show that π_0 is bounded.

Let $T = \sum_x A_x U_x \otimes l_x \in \mathfrak{C}_0$. We will show that $\|\sum A_x U_x\| \leq \|T\|$ (thus, $\|\pi_0\| \leq 1$). Fix $\varepsilon > 0$, and let F be a finite subset of G such that $A_x = 0$ for all $x \notin F$. By Lemma 2.2 (iii), there exists a unit vector $\xi \in l^2(G)$ such that $|(l_x \xi, \xi) - 1| \leq \varepsilon$, $x \in F$. For $f, g \in \mathcal{H}$, $\|f\| = \|g\| = 1$, one has

$$(T(f \otimes \xi), g \otimes \xi) = \sum_x (A_x U_x f \otimes l_x \xi, g \otimes \xi) = \sum_x (A_x U_x f, g)(l_x \xi, \xi).$$

So the distance between this complex number and $\sum (A_x U_x f, g) = (\sum A_x U_x f, g)$ is not greater than

$$\sum |(A_x U_x f, g)| \cdot |(l_x \xi, \xi) - 1| \leq \varepsilon \sum \|A_x U_x\|.$$

Hence, $$|(\sum A_x U_x f, g)| \leq |(T(f \otimes \xi), g \otimes \xi)| + \varepsilon \sum \|A_x U_x\| \leq \|T\| + \varepsilon \sum \|A_x U_x\|.$$

Since ε is arbitrary, we have $|(\sum A_x U_x f, g)| \leq \|T\|$, and the proof is completed by taking the supremum over $\|f\| = \|g\| = 1$.

Conversely, suppose Φ is faithful. We show that G satisfies condition (ii) of Lemma 2.2. Let $\lambda \in \mathbf{C}_{00}(G)$ be such that $\sum \lambda(x) l_x \geq 0$. Now by Lemma 2.4, π_0 extends to a *-homomorphism π of $\mathfrak{C}$ into $\mathcal{B}$. Hence, $\pi \circ \sigma^{-1}$ is a *-homomorphism of $[[l_x : x \in G]]$ into $\mathcal{B}$; in particular, $T = \pi \circ \sigma^{-1}(\sum \lambda(x) l_x)$ is a positive operator in $\mathcal{B}$. We have,

$$T=\pi(\sum \lambda(x)\,\sigma^{-1}(l_x))=\pi(\sum \lambda(x)\,U_x\otimes l_x)=\sum \lambda(x)\,U_x.$$

Letting f_0 be a unit vector such that $U_x f_0=f_0$ for all x, we have $(U_x f_0, f_0)=1$ for all x, and hence

$$\sum \lambda(x)=\sum \lambda(x)\,(U_x f_0, f_0)=(Tf_0, f_0)\geqslant 0.$$

Thus, G satisfies condition (ii), and the proof is finished.

Remarks. It seems curious that this property of Φ is determined by the algebraic structure of G and has nothing to do with the properties of the particular free action at hand. Thus, if G is amenable, the Φ based on any m-preserving free action of G is faithful; if G is not amenable, the Φ based on any such action of G fails to be faithful.

As an application of Theorem 2.6, one has the following generalization of Lemma 1.6.

COROLLARY. *Suppose G is amenable. Let S be a sub semigroup of G such that $S\cap S^{-1}=\{e\}$, and let $\mathcal{A}(S)$ be the Banach algebra generated by M and the unitary semigroup $\{U_x : x\in S\}$. Then Φ is multiplicative on $\mathcal{A}(S)$, and $\mathcal{A}(S)\cap\mathcal{A}(S)^*=M$.*

Proof. The proof is an imitation of the corresponding results of § 1. For example, if $x, y\in S$, then $\Phi(U_x U_y)=\Phi(U_{xy})=0$ or I according as $xy\neq e$ or $x=y=e$ (since $S\cap S^{-1}=\{e\}$). It follows easily that Φ is multiplicative on $\mathcal{A}(S)$. By Theorem 2.6, Φ is faithful, and one can now prove $\mathcal{A}(S)\cap\mathcal{A}(S)^*=M$ as in Lemma 1.6.

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